Exact solution of scalar field cosmology with exponential potentials and transient acceleration

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We show that the general solution of scalar field cosmology in d dimensions with exponential potentials for flat Robertson-Walker metric can be found in a straightforward way by introducing new variables which completely decouple the system. The explicit solution shows the region of parameters where the expansion has eternal acceleration, transient periods of acceleration, or no period of acceleration at all. In the cases of transient acceleration, the energy density exhibits a plateau during the accelerated expansion, where $p \cong -\rho$, due to dominance of potential energy. We determine the interval of accelerated expansion in terms of a simple formula. In particular, it shows that the period of accelerated expansion decreases in higher dimensions.

Understanding the origin of dark energy in the context of fundamental theories such as M/string theory became an important problem, after the recent WMAP data [1] confirming that nearly 70% of the energy density of the universe is in the form of dark matter. The dimensional reduction of these fundamental theories to four dimensions typically gives rise to scalar fields with exponential potentials coupled to four-dimensional gravity. Given that a substantial fraction of the energy density of the universe might indeed consist of quintessence [2] in the form of a slowly-rolling scalar field, it is of interest to understand whether exponential potentials could describe observational data for the late-time cosmic acceleration.

Exponential potentials in four dimensions were much investigated in the past (see e.g. [3]), and several exact solutions have appeared (for a recent discussion and further references, see [4]). In particular, a general solution in four dimensions was obtained in [5] and more recently in [6], and previous discussions of solutions in d dimensions can be found in [7]. Here we will show that the simplest case of a homogeneous scalar field coupled to an exponential potential can be actually solved in a direct and straightforward way in d dimensions by the introduction of new variables which decouple the system. The resulting general solution expressed in a suitable time frame is remarkably simple. The presence of exponential potentials in higher dimensional theories obtained from fundamental theories is quite generic, so the d dimensional case is of particular interest to compare cosmology in different dimensions, and to see to what extent some physical properties are generic or peculiar to four dimensions.

For four-dimensional cosmologies with exponential potentials, some new interesting aspects were recently pointed out in [8]. There it was shown an example of cosmology which starts with a decelerating expansion, at some point it experiences a transitory period of acceleration, and it ends with decelerating expansion again. The origin of the acceleration was further clarified in [9] (related discussions and generalizations can be found in [10]). The explicit general solution presented here gives a clear view of the window of parametric space where there is such period of accelerated expansion, and the dependence of this period on the dimensionality of the space.

We start with the action for 4d Einstein gravity coupled to a scalar field with an exponential potential V:

$$S = \int d^4x\sqrt{-g}\Big(\frac{R}{2\kappa^2} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V_0\exp(-\lambda\phi)\Big) \tag{1}$$

We will work in units where $\kappa^2 = 8\pi G_N = 1/2$. Here $V_0 > 0$, $\lambda > 0$ (the case $\lambda < 0$ is connected to the case $\lambda > 0$ by the change $\phi \to -\phi$).

Consider homogeneous and isotropic cosmologies described by the Robertson-Walker metric for a flat universe $(k=0)$ $ds^2 = -dt^2 + a^2(t)dx^i dx^i,\quad i = 1,2,3$. The action takes the form

$$S = \int d^3x dt\Big(- 6a\dot{a}^2 + a^3\big(\frac{1}{2}\dot{\phi}^2 - V_0\exp(-\lambda\phi)\big)\Big) \ . \tag{2}$$

The equations of motion are

$$\ddot{\phi} + 3H\dot{\phi} - \lambda V_0 e^{-\lambda\phi} = 0 \ , \tag{3}$$

$$H^2 = \frac{1}{6}\big(\frac{1}{2}\dot{\phi}^2 + V_0\exp(-\lambda\phi)\big) \ , \qquad H = \frac{\dot{a}}{a} \ . \tag{4}$$

For $V_0 > 0$, they have the late-time attractor solution:

$$\phi = \frac{2}{\lambda}\log(\frac{t}{t_0}) \ , \quad a = a_0(\frac{t}{t_0})^p, \quad p = \frac{1}{\lambda^2} \ ,$$
$$\lambda^2 V_0 t_0^2 = 2(3p-1) \ . \tag{5}$$

The solution exists provided $3p > 1$, i.e. $\lambda < \sqrt{3}$. The solution (5) shows that the expansion exhibits eternal acceleration when $\lambda < 1$.

Now we perform a change of variables $\{a(t), \phi(t)\} \to \{u(t), v(t)\}$ defined as follows:

$$\phi = \frac{1}{\kappa}\sqrt{\frac{2}{3}}(v-u) \ , \qquad a^3 = e^{v+u} \ . \tag{6}$$

The action takes the form

$$S = -\int d^3x dt \ e^{u+v}\left(\frac{8}{3}\dot{u}\dot{v} + V_0 e^{-2\alpha(v-u)}\right) \ , \tag{7}$$
$$\alpha \equiv \frac{\lambda}{\sqrt{3}} \ .$$

Next, we introduce a new time coordinate τ:

$$\frac{d\tau}{dt} = \sqrt{\frac{3V_0}{8}} \ e^{\alpha(u-v)} \ . \tag{8}$$

The action becomes

$$S = -\sqrt{\frac{8V_0}{3}}\int d^3x d\tau \ e^{u+v}e^{\alpha(u-v)}\big(u'v'+1\big) \ , \tag{9}$$

where prime denotes derivative with respect to τ. The equations of motion become decoupled,

$$\begin{aligned} v'' + (1-\alpha){v'}^2 - 1 - \alpha &= 0 \ , \\ u'' + (1+\alpha){u'}^2 - 1 + \alpha &= 0 \ , \end{aligned} \tag{10}$$

and the Hamiltonian constraint reduces to

$$v'u' = 1 \ . \tag{11}$$

Let us first consider the special case $\alpha = 1$. This corresponds to the limiting value for the attractor solution $\lambda = \sqrt{3}$. The direct integration of (10) gives

$$u = \frac{1}{2}\log(2\tau) \ , \qquad v = \tau^2 \ , \tag{12}$$

where the integration constants have been removed by rescaling the coordinates x_i, by a shift in time and by a shift in the scalar field. Thus

$$\begin{aligned} ds^2 &= \frac{8}{3V_0}\frac{e^{2\tau^2}}{2\tau}d\tau^2 + e^{\frac{2\tau^2}{3}}(2\tau)^{\frac{1}{3}}dx^i dx^i \ , \\ \phi &= \frac{1}{\sqrt{3}}\big(2\tau^2 - \log(2\tau)\big) \ . \end{aligned} \tag{13}$$

At late times, this has the behavior $a \sim t^{1/3}$ and $\phi \sim \frac{2}{\sqrt{3}}\log t$, which coincides with the attractor solution in the limit $\lambda = \sqrt{3}$.

The case $\alpha = 1/2$ can be studied directly in terms of the original time coordinate t, by the action (7), introducing variables $X = e^u$, $Y = e^v$. Then we have the equations $\ddot{X} = 0$, $\ddot{Y} = \frac{3}{4}V_0 X$, which are readily solved. This leads to

$$a^3 = \frac{1}{8}V_0 t^4 + c_1 t \ , \quad \phi = \frac{2}{\sqrt{3}}\log\big(\frac{1}{8}V_0 t^2 + \frac{c_1}{t}\big) \ . \tag{14}$$

Let us now consider the general solution to the system (10). It is convenient to discuss separately the cases $\alpha < 1$ and $\alpha > 1$.

a) Case $\alpha < 1$ ($\lambda < \sqrt{3}$):

$$u = \sqrt{\frac{1-\alpha}{1+\alpha}}\ \tau + \frac{1}{1+\alpha}\log\left(1 - me^{-2w\tau}\right) \tag{15}$$

$$v = \sqrt{\frac{1+\alpha}{1-\alpha}}\ \tau + \frac{1}{1-\alpha}\log\left(1 + me^{-2w\tau}\right) \tag{16}$$

with $w \equiv \sqrt{1-\alpha^2}$. Thus the metric and scalar field are

$$\begin{aligned} ds^2 &= -\frac{8}{3V_0} e^{\frac{4\alpha^2\tau}{w}} \frac{(1+me^{-2w\tau})^{\frac{2\alpha}{(1-\alpha)}}}{(1-me^{-2w\tau})^{\frac{2\alpha}{(1+\alpha)}}} d\tau^2 + e^{\frac{4\tau}{3w}} (1+me^{-2w\tau})^{\frac{2}{3(1-\alpha)}} (1-me^{-2w\tau})^{\frac{2}{3(1+\alpha)}} dx^i dx^i \\ \phi &= \frac{2}{\sqrt{3}}\left(\frac{2\alpha\tau}{w} - \frac{1}{1+\alpha}\log(1-me^{-2w\tau}) + \frac{1}{1-\alpha}\log(1+me^{-2w\tau})\right) . \end{aligned} \tag{17}$$

For $m \neq 0$ the absolute value of m can be absorbed into a shift of τ. The attractor solution is the particular case $m = 0$, and it is the asymptotic limit of the whole family of solutions. Indeed, for large τ, we have $t = \sqrt{\frac{8}{3V_0}}\frac{w}{2\alpha^2}\exp(2\alpha^2\tau/w)$. Substituting into the scalar field ϕ and into the scale factor a^2, we reproduce the attractor solution (5).

Note that the solution (17) also includes the case when $\lambda = 0$, corresponding to a cosmological constant. In this case $\tau = \sqrt{\frac{3V_0}{8}}\ t$, and the general solution is:

$$a^3 = a_0^3 e^{2\tau}(1 - m^2 e^{-4\tau}) \ , \qquad \phi = \frac{2}{\sqrt{3}}\log(\frac{1+me^{-2\tau}}{1-me^{-2\tau}})$$

When $m = 0$, we recover the standard de Sitter solution with $\phi =$constant and $a = a_0 e^{2\tau/3} = a_0 e^{\sqrt{\frac{V_0}{6}}t}$.

b) Case $\alpha > 1$ ($\lambda > \sqrt{3}$):

The general solution is given by

$$u = \frac{1}{\alpha+1}\log\left[\sin\beta\right] \ , \qquad v = -\frac{1}{\alpha-1}\log\left[\cos\beta\right] \tag{18}$$

where $\beta \equiv \sqrt{\alpha^2-1}\ \tau + \beta_0$. Thus

$$\begin{aligned} ds^2 &= -\frac{8}{3V_0}(\cos\beta)^{-\frac{2\alpha}{\alpha-1}}(\sin\beta)^{-\frac{2\alpha}{\alpha+1}} d\tau^2 + \frac{(\sin\beta)^{\frac{2}{3(\alpha+1)}}}{(\cos\beta)^{\frac{2}{3(\alpha-1)}}} dx^i dx^i \ , \\ \phi &= -\frac{2}{\sqrt{3}}\left(\frac{1}{\alpha+1}\log(\sin\beta) + \frac{1}{\alpha-1}\log(\cos\beta)\right). \end{aligned} \tag{19}$$

Let us now consider the equation of state at early and late times. We first consider the case $m > 0$ and $\alpha < 1$. Setting $m = 1$, time $t \cong 0$ corresponds to τ with $1 \cong e^{-2w\tau}$. In the vicinity of this point, eqs. (17) imply

$$t \sim \tau^{\frac{1}{\alpha+1}} \ , \quad a^3 \sim \tau^{\frac{1}{\alpha+1}} \ , \quad \phi = -\frac{2}{\sqrt{3}(1+\alpha)}\log 2w\tau \ , \tag{20}$$

i.e. $a \sim t^{1/3}$. This corresponds to the kinetic attractor, with the equation of state $p = \rho$ corresponding to matter with dominance of kinetic energy. The same result $a \sim t^{1/3}$ holds for $m < 0$. This is clear from eqs. (15), (16) since the change $m \to -m$ is equivalent to changing $\alpha \to -\alpha$ and $u \to v$, which leaves the metric invariant. For large t, one has the standard result $a \sim t^p$ that follows from the late-time attractor solution, corresponding to $p = \omega\rho$, $\omega = 2\alpha - 1 = \frac{2}{3}\lambda^2 - 1$.

The cases that arise from dimensional reduction to four dimensions on hyperbolic space $\mathcal{H}_n$ (discussed in [8, 9, 10]) correspond to a discrete set of λ of the form $\lambda = \sqrt{(n+2)/n}$, with $n = 2, 3...$. In these cases, the asymptotic behavior $a \sim t^p$ has a simple interpretation as arising from higher dimensional Milne space [11].

Fig. 1 shows a plot of the energy density as a function of τ for $\alpha = 2/3$ and $m > 0$, which exhibits the generic behavior for any α in the interval $(1/\sqrt{3}, 1)$. The kinetic energy vanishes at a certain time; this is the instant where ϕ is reflected off the potential barrier, as observed in [9]. We see that the energy density exhibits a plateau during the period of acceleration, in agreement with the fact that near $\dot{\phi} = 0$ the equation of state is $p \cong -\rho$ so that $\dot{\rho} = -3H(\rho + p) \cong 0$. The potential as a function of time intersects the curve representing twice the kinetic energy at two points. Since one of the Einstein equations imply that R_{00} is proportional to $2T - V$, the period between these two points is precisely the period of accelerated expansion.

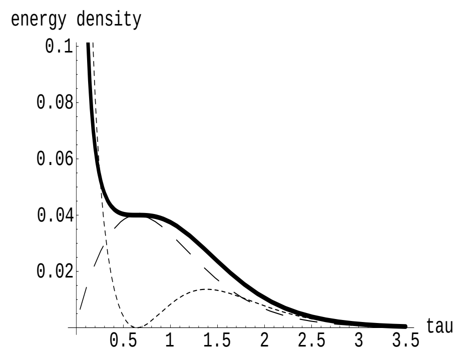

FIG. 1: Energy density as a function of time τ for $\alpha = 2/3$ (solid line). The period of accelerated expansion can be visualized as the interval between the intersection of twice the kinetic energy (short-dashed line) and the potential energy (long-dashed line). Here $\alpha = 2/3$ and the same qualitative behavior appears for all $\alpha \in (1/\sqrt{3}, 1)$, $m > 0$.

Now consider the case $\alpha > 1$, where the solutions are given by eq. (19). The geometry has a singularity at $\beta = 0$ and $\beta = \frac{\pi}{2}$. Setting $\beta_0 = 0$ by a shift in τ, the time between the two singularities corresponds to the interval $\tau \in \big(0, \pi/(2\sqrt{\alpha^2 - 1})\big)$. By integrating $\frac{dt}{d\tau}$, we see that this in turn corresponds to a *infinite* proper time interval $t \in (0, \infty)$. In the vicinity of the two singularities, the geometry has the same behavior (20). Computing $\dot{a}$ one finds that it is positive definite, so the geometry describes an expanding universe, which starts and ends with a decelerating expansion.

For $\ddot{a}$ we find an equation similar to (30), with $Z \equiv \cos(2\sqrt{\alpha^2 - 1})\tau)$. The two roots are given by eq. (31). For $\alpha > 1$ one has $|Z_{1,2}| < 1$, so that the equation $Z_{1,2} \equiv \cos(2\sqrt{\alpha^2 - 1})\tau_{1,2})$ always has real solutions $\tau_{1,2}$ in the interval $0 < \sqrt{\alpha^2 - 1}\tau_{1,2} < \frac{\pi}{2}$, with $\tau_1(\alpha) < \tau_2(\alpha)$. As a result, $\ddot{a}$ is positive in the interval between τ_1 and τ_2. Thus solutions with $\alpha > 1$ also exhibit a transitory period of acceleration.

We now consider the generalization to d dimensions:

$$S = \int d^d x \sqrt{-g} \Big(\frac{R}{2\kappa^2} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \Big) \tag{21}$$

Setting $\kappa^2 = 8\pi G_N = 1/2$, and considering again the Robertson-Walker metric for a flat universe ($k = 0$)

$$ds^2 = -dt^2 + a^2(t) dx^i dx^i \ , \qquad i = 1, ..., d-1 \tag{22}$$

the action takes the form

$$S = \int d^d x \Big(-(d-1)(d-2) a^{d-3} \dot{a}^2 + a^{d-1} \big(\frac{1}{2} \dot{\phi}^2 - V(\phi) \big) \Big)$$

Now we need to introduce the analogous u, v variables. They are defined as follows:

$$\phi = \frac{1}{\kappa} \sqrt{\frac{d-2}{d-1}} (v - u) \ , \qquad a^{d-1} = e^{v+u} \tag{23}$$

The action becomes

$$\begin{aligned} S &= -\int d^{d-1}x dt \ e^{u+v} \Big(\frac{2(d-2)}{\kappa^2 (d-1)} \dot{u}\dot{v} + V_0 e^{-2\alpha(v-u)} \Big), \\ \alpha &\equiv \frac{1}{2\kappa} \sqrt{\frac{d-2}{d-1}} \ \lambda \ . \end{aligned} \tag{24}$$

Introduce a new time variable τ

$$\frac{d\tau}{dt} = \kappa\sqrt{\frac{(d-1)V_0}{2(d-2)}}\ e^{\alpha(u-v)} \tag{25}$$

The action becomes

$$S = -\frac{1}{\kappa}\sqrt{\frac{2(d-2)V_0}{d-1}}\int d^{d-1}x d\tau\ e^{u+v}e^{\alpha(u-v)}\big(u'v'+1\big)\ . \tag{26}$$

Thus we get the same system for all dimensions. The solutions for u and v are the same as before. In particular, the general solution is given by eqs. (15), (16) for $\alpha<1$, and by eqs. (18) for $\alpha>1$. The metric and the scalar field are then read from (23) and (25). For $\alpha<1$:

$$\begin{aligned} ds^2 &= -\frac{2(d-2)}{\kappa^2(d-1)V_0}\ e^{\frac{4\alpha^2\tau}{w}}\frac{(1+me^{-2w\tau})^{\frac{2\alpha}{(1-\alpha)}}}{(1-me^{-2w\tau})^{\frac{2\alpha}{(1+\alpha)}}}d\tau^2 + e^{\frac{4\tau}{sw}}(1+me^{-2w\tau})^{\frac{2}{s(1-\alpha)}}(1-me^{-2w\tau})^{\frac{2}{s(1+\alpha)}}dx^i dx^i \\ \phi &= \frac{1}{\kappa}\sqrt{\frac{d-2}{d-1}}\bigg(\frac{2\alpha\tau}{w} - \frac{1}{1+\alpha}\log(1-me^{-2w\tau}) + \frac{1}{1-\alpha}\log(1+me^{-2w\tau})\bigg)\ , \end{aligned} \tag{27}$$

with $s=d-1$. The solution has the following behavior:

$$\begin{aligned} &a\sim t^{\frac{1}{d-1}}\ , \quad \phi = -\frac{1}{\kappa}\sqrt{\frac{d-2}{d-1}}\log t\ , \quad \text{for} \quad t\cong 0\ , \\ &a\sim t^{\frac{4\kappa^2}{(d-2)\lambda^2}}\ , \quad \phi = \frac{2}{\lambda}\log t\ , \qquad \text{for} \quad t\gg 1\ . \end{aligned} \tag{28}$$

The equation of state at initial times is $p=\rho$, while at late times

$$p=\omega\rho, \qquad \omega = \frac{d-2}{2\kappa^2(d-1)}\lambda^2 - 1\ . \tag{29}$$

The geometry describes an expanding cosmology. According to eq. (28), the expansion has eternal acceleration if $\lambda < \frac{2\kappa}{\sqrt{d-2}}$. We now show that when $\frac{2\kappa}{\sqrt{d-2}} < \lambda < 2\kappa\sqrt{\frac{d-1}{d-2}}$, the solution (27) with $m>0$ has a transient period of acceleration, just as the $d=4$ solution. Computing $\frac{da}{dt}=\frac{da}{d\tau}\frac{d\tau}{dt}$ from (17), one sees that $\dot a$ is proportional to a quantity which is positive definite for all m and τ. Setting $|m|=1$ by a shift of τ and computing $\ddot a$, we obtain:

$$\ddot a = -(\text{positive})\Big(\big((d-1)\alpha^2-1\big)Z^2 - 2(d-2)\,\text{sign}(m)\alpha Z + d-1-\alpha^2\Big) \tag{30}$$

with $Z\equiv\cosh(2w\tau)$. If $(d-1)\alpha^2<1$ (corresponding to $\lambda<\frac{2\kappa}{\sqrt{d-2}}$), we get eternal acceleration for all m, since at late times the first term dominates. This is expected, in consistency with the attractor solution. If $(d-1)\alpha^2>1$, at late times the solution always describes a decelerating expansion. If $(d-1)\alpha^2>1$ and $m<0$, then the right hand side of (30) is negative definite, implying deceleration at all times. Finally, in the case $(d-1)\alpha^2>1$ and $m>0$, the solution always exhibits the transient periods of acceleration noticed in [8]. Indeed, eq. (30) has two roots

$$Z_\pm = \frac{(d-2)\alpha \pm \sqrt{d-1}(1-\alpha^2)}{(d-1)\alpha^2-1}\ . \tag{31}$$

This determines the interval $\tau_-(\alpha)<\tau<\tau_+(\alpha)$ of accelerated expansion. The roots $\tau_\pm$ are real (and positive), since in the relevant interval $\alpha\in(1/\sqrt{d-1},1)$ we have $Z_\pm>1$. In the limit $\alpha\to 1$, the two roots coincide, and the period of accelerated expansion goes to zero. In the opposite limit, $\alpha\to 1/\sqrt{d-1}$, one gets $Z_-=d/(2\sqrt{d-1})$ and $Z_+\to\infty$, so there is an infinite period of acceleration. For large dimensions, one gets $Z_\pm = 1/\alpha \pm (1-\alpha^2)/(\alpha^2\sqrt{d}) + O(1/d)$, which shows that the transient period of acceleration is shorter in higher dimensions.

In conclusion, we have shown that scalar field cosmology with an exponential potential can be solved in a simple way in any spacetime dimension by the introduction of suitable variables which completely decouple the system. We have given the explicit solution in d dimensions, which exhibits initial and late time attractors, and determine the corresponding equations of state. The mechanism of transient acceleration observed in [8] subsists in higher dimensions. This could be of relevance for string/M-theory cosmological scenarios for early universe where d dimensions are

comparable and much larger than the Planck length, so that the effective field theory description applies. In four dimensions, as discussed in [6], the effect of transient acceleration could explain the observed accelerated expansion of the universe, but it seems insufficient to model early universe inflation. However, as pointed out above, the period of accelerated expansion can be very large for exponential potentials with $\alpha \sim 1/\sqrt{3}$. This effect could also apply in cosmologies with more general potentials exhibiting periods of transient acceleration, i.e. there could be critical values of the parameters where the acceleration period tends to infinity. In order to describe inflation within an effective field theory approach, it seems worthwhile to search for string theory compactifications that could give rise to such potentials which are close to these critical potentials.

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